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HARMONIC ANALYSIS FROM QUASI-PERIODIC DOMAINS

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ABSTRACT

Harmonic analysis is applied in a quasi-periodic context to get rigidity results in orbit equivalence theory.

1. Introduction

In his paper "Random walk on random groups" Gromov devised a set of tools to establish general fixed point theorems for finitely generated groups acting isometrically on non-positively curved spaces [19]. The underlying techniques, which have their origins in harmonic analysis, are aimed at deriving the existence of fixed points from a local property of the acting group called the '$\lambda_1 > 1/2$ criterion'.

The purpose of the present paper is to develop these tools in a measure theoretic dynamical framework. Our original motivation stems from the construction of quasi-periodic affine triangle buildings and the existence of minimal sublaminations of the space of triangle buildings [3, 4]. Affine triangle buildings are typical examples of polyhedra satisfying the $\lambda_1 > 1/2$ local criterion.

As is well-known since the work of Eells-Sampson [11], variational principles (consisting of minimizing certain energy functionals, usually along the heat flow) lead to the existence of harmonic mappings between Riemannian manifolds when both are compact and the target has non-positive sectional curvature.

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This approach has proved to be very flexible. Harmonicity extends to a variety of contexts and makes sense for mappings between more general spaces, allowing (simplicial) singularities, equivariance with respect to group actions, etc. This applies, for instance, to the superrigidity of lattices, especially in the rank one and the p-adic cases, and provides geometric approaches to Margulis' celebrated superrigidity theorem for higher rank lattices (at least in the uniform case). See [20, 12] and the references therein for more details.

The $\lambda_1 > 1/2$ criterion originated in the work of Garland [16], who proved a remarkable vanishing theorem for the low dimensional cohomology of lattices in higher rank p-adic groups. See [16, 6, 7, 34, 2, 25, 31, 32, 19, 17, 30, 35, 28, 21] for subsequent developments. The most recent results, in which we are presently interested, consist of general **non-linear fixed point theorems** for various kinds of '$\lambda_1 > 1/2$ groups' acting on various kinds of non-positively curved spaces (see, in particular, [31, 32, 19, 21]).

We are primary concerned with measured dynamical systems and measured equivalence relations. In the early eighties the rigidity of lattices in higher rank Lie groups has been recognized at the level of their probability measure preserving actions on standard Borel spaces via Zimmer's famous cocycle superrigidity theorem. This phenomenon, which exhibits strong **orbit equivalence rigidity** properties of certain measure preserving actions, received much attention in the last five years with the fundamental work of Furman, Gaboriau, Monod-Shalom, and Popa (see, for instance, [14, 15, 24, 29]).

Before going any further let us recall some classical concepts in orbit equivalence theory and measured equivalence relations [13, 9, 15]. Let X be a standard Borel space, R an equivalence relation on X with countable classes. One says that R is Borel if its graph $R \subset X \times X$ (the couples of equivalent points) is a Borel subset of $X \times X$. Given two Borel equivalence relations R and R' with countable classes on standard Borel spaces X and X' respectively, one says that a Borel map $\pi : X \to X'$ is an homomorphism from R to R' if $x \sim_R y$ implies that $\pi(x) \sim_{R'} \pi(y)$. A Borel equivalence relation on X with countable classes is called a measured equivalence relation if X is endowed with a quasi-invariant probability measure. Quasi-invariance means that the saturation of a negligible Borel subset of X is still negligible (if in addition the measure of every Borel subset is preserved by the dynamic, then the measure is called invariant). The graph of a measured equivalence relation is endowed with the canonical measure class for which the projection onto the first component is non-singular.

A graphing of a Borel equivalence relation R (with countable classes) is a Borel subset $K \subset R$ such that the increasing union $\bigcup_n K^n$ is equal to R, where $K^n = \{(x_0, x_n) \in R : \exists x_1, \ldots, x_{n-1} \in X,\ (x_i, x_{i+1}) \in K\}$ is the n-th convolution product of K. In this case each class of R is endowed with a **connected** graph structure, where by definition an edge is present between $x, y \in X$ if and only if $(x, y) \in K$. A graphing is called a triangulation if the minimal cycles in these graphs all have length 3. Given a triangulation K, the link L_x at a vertex $x \in X$ is the sphere of radius 1 centered at x in the graph structure on the equivalence class of x. This definition extends to measured equivalence relations with the usual 'up to a negligible set' decorations. In this case one defines the volume of K to be the average number (with respect to the given probability measure on X) of non-oriented simplexes in K. (Compare with Section 3.)

Let L be a finite connected graph and Y be a metric space. One defines $\lambda_1(L, Y)$ to be the largest constant λ such that the following inequality

$$\frac{1}{2\tau} \sum_{u,v \in L^{(0)}} |f(u) - f(v)|^2 \tau(u)\tau(v) \leq \frac{1}{\lambda} \sum_{(u,v) \in L^{(1)}} |f(u) - f(v)|^2$$

holds true for any map $f : L^{(0)} \to Y$, where $\tau(u)$ is the valence of a vertex $u \in L^{(0)}$, τ is the number of edges in L, and $|y - z|$ is the distance between y and z in Y. This constant has been introduced by Gromov and the case $Y = \mathbf{R}$ gives back the usual definition of $\lambda_1(L)$ as the first non-zero eigenvalue of the Laplace operator on L, see [19] (see also [32]).

We are interested in measured equivalence relations admitting a triangulation almost all of whose links L_x satisfy the condition $\lambda_1(L_x) > 1/2$, uniformly in $x \in X$. As mentioned above the space of triangle buildings considered in [3, 4] provides non-trivial examples of ergodic measured equivalence relations satisfying this condition (note that the existence of non-atomic invariant probability measures on this space is not known yet). Recall that a measured equivalence relation on X is said to be ergodic if every invariant measurable subset of X has measure 0 or 1.

Let us first state two corollaries (see Section 6) of the fixed point theorem proved below (Theorem 3). The definition of CAT(0) spaces is recalled in Section 2.

Setup for the corollaries. Fix a standard probability space (X, μ), an ergodic equivalence relation R on X, and assume that the probability measure

μ is invariant with respect to R. Consider a triangulation K of R with finite volume (with respect to μ) and almost everywhere connected links, and fix a surjective Borel map $\pi : X \to X'$ which is an homomorphism from R to R'.

Corollary 1: *Assume that the space X' is a Borel transversal of a lamination F on a compact space M, all of whose leaves are simply connected manifolds with non-positive sectional curvature. If almost every link L_x of K satisfies the condition $\lambda_1(L_x, \ell) \geqslant \lambda$ for all leaves ℓ of F and a constant $\lambda > 1/2$, and if the following finite energy condition is satisfied*

$$\int_X \sum_{(x,y)\in K} |\pi(y) - \pi(x)|^2 d\mu(x) < \infty,$$

then the μ-essential image of π is contained in a single leaf of F.

The "finite energy condition" is satisfied for instance at the topological level, when X is a compact topological space and the map π is sufficiently continuous. This corollary roughly says that there is no (finite energy) **smoothing** of an equivalence relation satisfying the assumptions (compare to [18]). Recall that a Borel subset X' of M is said to be a transversal of F if its intersection with every leaf of F is a countable set, consisting of a class of R'.

Corollary 2: *Assume that we are given a CAT(0) space Y such that $\lambda_1(L, Y) \geqslant \lambda$ for almost all links L of K and a real number $\lambda > 1/2$, not depending on L, and a finitely generated group Γ which is orbit equivalent to R'. Thus we have an isomorphism $\Gamma \ltimes X' \simeq R'$ and a corresponding cocycle c from R' to Γ. If the following finite energy condition is satisfied,*

$$\int_X \sum_{(x,y)\in K} |c(\pi(x,y))|_\Gamma^2 d\mu(x) < \infty,$$

then any isometric action of Γ on Y has a fixed point.

Here the metric $|\cdot|_\Gamma$ in Γ is the word metric taken with respect to any finite generating set. Corollary 2, in particular, restricts the set of possible actions of Γ that are orbit equivalent to R.

We now state the fixed point theorem on CAT(0) spaces, without elaborating much on the terminology (the precise definitions will be given later on). The global picture is as follows. There is a 'quasi-periodic' source space Σ with a diffusion ν and a CAT(0) target Y, both are (fibered spaces) endowed with an action of a discrete measured groupoid G. Under the assumptions the heat flow, which is defined on the space of G-equivariant mappings from Σ to Y with finite energy, converges to a fixed point relative to ν.

THEOREM 3: *Let G be a discrete measured groupoid and (Σ, λ, ν) be a G-quasi-periodic symmetric diffusion space. Any complete CAT(0) G-space Y for which there is a finite energy equivariant map $\mathfrak{f} : \Sigma \to Y$, and which satisfies the (Poincaré type) condition $\pi_2^{\mathfrak{f}}(\Sigma, Y) < 2$, has a fixed point relative to ν at finite L^2-distance from $\mathfrak{f}$.*

This theorem is due to M. Gromov in the case of countable discrete groups (see [19, page 125]). Using an “integral geometry argument” à la Garland, he then deduced a fixed point theorem for 2-dimensional simplicial complexes Σ satisfying the $\lambda_1 > 1/2$ local criterion. This fixed point theorem has been obtained independently by Izeki and Nayatani [21] and corresponds to Theorem 18 in the present paper. We mention that Izeki and Nayatani are using a definition of λ_1 introduced by Wang [32], different from the one above and which behaves better with respect to (some) tangent cones. The first non-linear fixed point theorem was proved by Wang in [31, 32]. The proof we give here follows Section 3 in [19] as carefully as possible, although we will in some places be content with technical arguments (other alternatives are probably laid down in [19]). As often with this type of result the proof confronts a deformation process to a local rigidity property. The linear case (when Y is a Hilbert space) was studied in [28], following [19, 17]. The “finite energy assumption” appearing in the theorem comes from the direct use of harmonic techniques: the existence of finite energy mapping $\Sigma \to Y$ is required in order to start the diffusion process (this assumption is always satisfied in the situation of [19, 21]).

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2. Metric spaces of non-positive curvature

A general reference for non-positively curved spaces is [5]. We shall only recall here the background material for the main theorems. We follow [19, Section 3]. The relevant property of the target space which allows to define the heat flow is the presence of a **codiffusion** from probability measures on this space back to the space itself. The terminology for the inequalities appearing below follows [19] as well.

We will use the following elementary estimate.

LEMMA 4: *For any non-negative real numbers* $a_1, \ldots, a_n$ *one has* $\left(\sum_{i=1}^n a_i\right)^2 \leq \sum_{i=1}^n a_i^2/t_i$ *where* $t_1, \ldots, t_n$ *are positive and satisfy* $\sum_{i=1}^n t_i = 1$.

Proof. Minimize the function $\sum_{i=1}^n a_i^2/t_i$ under the constraint $\sum_{i=1}^n t_i = 1$ (Lagrange multipliers). ∎

2.1. CONVEXITY PROPERTIES OF GEODESIC METRICS. Let Y be a complete metric space with metric d. Recall that d is said to be **geodesic** if any two points of Y can be joined by a geodesic segment in Y, that is, an isometric embedding of some interval of the real line into Y. One says that Y is a **CAT(0) space** if d is geodesic and satisfies the following inequality.

CAT(0) INEQUALITY. For any point y of Y, any geodesic segment γ in Y with end points $y_0, y_1 \in Y$, and any point y_t at a fraction $t \in [0, 1]$ from y_0 to y_1 on γ, one has

$$|y - y_t|^2 \leq (1-t)|y - y_0|^2 + t|y - y_1|^2 - t(1-t)|y_0 - y_1|^2, \tag{1}$$

where we note $d(y_0, y_1) = |y_1 - y_0|$.

The inequality shows that there is a unique geodesic segment $[u, v]$ between any two points u, v, and that the map $|y - \cdot\,|^2$ is strictly convex on this segment. Hence there exists a unique point on any geodesic segment (or line) realizing the distance to a given point y. Examples of CAT(0) spaces range from (the most negatively curved) trees to (flat) Hilbert spaces, including complete simply connected Riemannian manifolds with non-positive sectional curvature and Bruhat-Tits buildings. Note that the CAT(0) inequality is an equality for Hilbert spaces (by taking coordinates it is sufficient to check it for the real line), and accordingly that a strict inequality indicates negative curvature. Any complete CAT(0) space is contractible to a point.

As described by the following inequality, the growth of the squared distance $|y - \cdot\,|^2$ to a given point $y \in Y$ along geodesic lines is actually parabolic.

2-CONVEXITY INEQUALITY. Let γ be a geodesic segment and $y_0 \in \gamma$ be the closest point to y. Then for $y_\ell \in \gamma$ at distance ℓ from y_0 one has

$$|y - y_\ell|^2 \geq \ell^2 + |y - y_0|^2. \tag{2}$$

Proof. This is immediate from the CAT(0) inequality (1) for the closest point $y_0 \in \gamma$ and $y_t \to y_0$:

$$0 \leq \frac{|y-y_t|^2 - |y-y_0|^2}{|y_0 - y_t|} \leq \frac{t}{|y_0 - y_t|}(|y-y_\ell|^2 - |y-y_0|^2) - |y_\ell - y_t|$$

where $|y_0 - y_t| = t|y_0 - y_\ell|$. ∎

2.2. BARYCENTRIC CODIFFUSION AND AVERAGED CONVEXITY INEQUALITIES. Let Y be a complete CAT(0) space.

Definition 5: A codiffusion on Y is a map c from the space P_2 of probability measure on Y with finite second moment to Y, such that $c(\delta_y) = y$ and $c^{-1}(y)$ is convex for any $y \in Y$.

Here P_2 is the (convex) space of probability measure ν on Y satisfying the condition

$$|\nu - y|^2 = \int_Y |y - z|^2 d\nu(z) < \infty$$

for any $y \in Y$.

Example 6: In the Hilbert case, the canonical codiffusion is the affine map $c(\nu) = \int_Y \xi d\nu(\xi)$.

Averaging the CAT(0) inequality with respect to $\nu \in P_2$,

$$|\nu - y_t|^2 \leq (1-t)|\nu - y_0|^2 + t|\nu - y_1|^2 - t(1-t)|y_0 - y_1|^2,$$

leads to the 2-convexity of the function $|\nu - \cdot\,|^2$ on geodesics, as for $|y - \cdot\,|^2$.

It is intuitively clear that a continuous functions f from Y to $\mathbf{R}_+$, which is 2-convex on every geodesic segments in Y (i.e. which satisfies the inequality (2) for a point y_0 realizing a minimum of f on γ), has a unique global minimum in Y. For a proof observe that the level sets $\{f \leq t\}$ of such a function all have bounded diameter, less than $2\sqrt{t-m}$, which goes to 0 as t goes to the infimum m of f, by 2-convexity.

The **barycentric codiffusion** is defined for any complete CAT(0) space Y in the following way. Let $\nu \in P_2$ be a probability measure on Y. The function $|\nu - \cdot\,|^2$ defined above has a unique minimizer $c(\nu) \in Y$, called the **center of mass** of ν. The barycentric codiffusion is the map $c : \nu \mapsto c(\nu)$. (For a more general probability measure $\nu \notin P_2$, the minimizer is not well-defined a priori, it might for instance fall into the boundary of Y.) Note that the barycentric codiffusion is equivariant with respect to the isometry group of Y.

Summing up, one has the following result.

Averaged 2-convexity inequality. Let $\nu \in P_2$ and γ be a geodesic segment. Denote by y_0 the closest point to ν on γ, which minimizes $|\nu - \cdot\,|^2$ on γ. Then for $y \in \gamma$ at distance ℓ from y_0 we have,

$$|\nu - y|^2 \geq \ell^2 + |\nu - y_0|^2. \tag{3}$$

This, for geodesics containing $c(\nu)$, reads

$$|\nu - y|^2 \geq |y - c(\nu)|^2 + |\nu - c(\nu)|^2 \tag{4}$$

for any $y \in Y$.

2.3. Averaged and polygonal inequalities. Integrating in y the averaged 2-convexity inequality (4) leads to the following inequality (henceforth referred to as a **Wirtinger inequality**),

$$|c(\nu) - \nu|^2 \leq \frac{1}{2} \int_{Y\times Y} |y - z|^2 d\nu(y) d\nu(z), \tag{5}$$

which for the uniform probability measure on 4 points gives, as an important particular case, the **quadrilateral inequality**: in any quadrilateral, the sum of the squared length of the diagonals is not greater than the sum of the squared length of the edges. More precisely,

$$|z_1 - z_3|^2 + |z_2 - z_4|^2 \leq |z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2 + |z_4 - z_1|^2, \tag{6}$$

for $z_1, z_2, z_3, z_4 \in Y$. This is clear if the 4 points are in the euclidean plane (as well as for Hilbert spaces) by taking coordinates. In Y this follows directly from 2-convexity, considering the measure $\nu = 1/2(\delta_{z_1} + \delta_{z_3})$, $\nu' = 1/2(\delta_{z_2} + \delta_{z_4})$, and integrate the ν-averaged 2-convexity inequality with respect to ν',

$$1/4(|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2 + |z_4 - z_1|^2) \geq 1/2 \sum_{i=1}^{4} |c(\nu) - z_i|^2.$$

As $|z_1 - z_3|^2 \leq 2(|c(\nu) - z_1|^2 + |c(\nu) - z_3|^2)$, this gives the result. Alternatively one can simply use the Wirtinger inequality directly with the uniform measure $\nu = 1/4 \sum_i \delta_{z_i}$, as stated above.

One deduces the continuity of the center of mass of $\nu \in P_2$ with respect to the displacement of the mass ν as follows. For $\nu, \nu' \in P_2$ define $\|\nu - \nu'\|_2^2 = \inf_\pi \int_{Y\times Y} |y-z|^2 d\pi(y,z)$, where the infimum is taken over the probability measure π on $Y \times Y$ with marginals ν and ν'. Then,

$$|c(\nu) - c(\nu')| \leq \|\nu - \nu'\|_2. \tag{7}$$

(barycentric contraction of the L^2 transportation metric). Indeed, integrating the quadrilateral inequality with respect to $d\pi(z_3, z_4)$ and putting $z_1 = c(\nu')$, $z_2 = c(\nu)$, we get

$$|c(\nu') - \nu|^2 + |c(\nu) - \nu'|^2 \leq |c(\nu) - c(\nu')|^2 + |c(\nu) - \nu|^2 + \|\nu - \nu'\|_2^2 + |c(\nu') - \nu'|^2,$$

thus the inequality follows from the averaged 2-convexity inequality (for $y = c(\nu), c(\nu')$).

Let us conclude this section by observing that the above quadrilateral inequality can be sharpened in the following way (these types of inequalities follow from the work of Reshetnyak, cf. Korevaar-Schoen [23, page 621]).

Quadrilateral inequality. Given four points z_1, z_2, z_3, z_4 in Y we have,

$$|z_1 - z_3|^2 + |z_2 - z_4|^2 \leq 2|z_1 - z_2||z_3 - z_4| + |z_2 - z_3|^2 + |z_4 - z_1|^2. \tag{8}$$

Proof. Take $\nu = (1-t)\delta_{z_1} + t\delta_{z_3}$ and $\nu' = (1-t)\delta_{z_2} + t\delta_{z_4}$ and integrate the ν-averaged 2-convexity inequality with respect to ν',

$$\begin{aligned}(1-t)^2|z_1 - z_2|^2 + (1-t)t|z_1 - z_4|^2 + (1-t)t|z_3 - z_2|^2 + t^2|z_4 - z_3|^2 \\ \geq (1-t)|c(\nu) - z_1|^2 + t|c(\nu) - z_3|^2 + (1-t)|c(\nu) - z_2|^2 + t|c(\nu) - z_4|^2.\end{aligned}$$

Using the fact that $(a+b)^2 \leq a^2/t + b^2/(1-t)$ for non-negative numbers a and b (Lemma 4) we get $(1-t)t|z_1 - z_3|^2 \leq (1-t)|c(\nu) - z_1|^2 + t|c(\nu) - z_3|^2$ (for $a = |c(\nu) - z_1|$, $b = |c(\nu) - z_3|$, $|z_1 - z_3| \leq a + b$). So

$$|z_1 - z_3|^2 + |z_2 - z_4|^2 \leq \frac{1-t}{t}|z_1 - z_2|^2 + |z_1 - z_4|^2 + |z_3 - z_2|^2 + \frac{t}{1-t}|z_4 - z_3|^2,$$

and as $(1-t)/t$ takes all positive values one can now set $(1-t)/t = \frac{|z_4 - z_3|}{|z_1 - z_2|}$ instead of 1. ∎

The previous proof of the barycentric contraction property works as well for the version (8) of the quadrilateral inequality and yields the following result.

Barycentric contraction of the L^1 transportation metric. For $\nu, \nu' \in P_2$ define $\|\nu - \nu'\|_1 = \inf_\pi \int_{Y\times Y} |y - z| d\pi(y, z)$, where the infimum is taken over the probability measure π on $Y \times Y$ with marginals ν and ν'. Then

$$|c(\nu) - c(\nu')| \leq \|\nu - \nu'\|_1. \tag{9}$$

Proof. Integrating the quadrilateral inequality with respect to $d\pi(z_3, z_4)$ and putting $z_1 = c(\nu')$, $z_2 = c(\nu)$ we get,

$$|c(\nu')-\nu|^2+|c(\nu)-\nu'|^2 \leq 2|c(\nu)-c(\nu')| \int |y-z|d\pi(y,z)+|c(\nu)-\nu|^2+|c(\nu')-\nu'|^2,$$

thus the inequality follows from the averaged 2-convexity inequality. ∎

3. Quasi-periodic spaces and isometric representations

We now describe the (quasi-periodic) source space, where the **diffusion** takes place. We assume all diffusions to be **symmetric** relative to a fixed covolume. Every diffusion has an **energy** associated to it, defined on mappings with values in some metric space. The rôle of the **Poincaré inequalities** is to relate the energies of differently spread diffusions.

3.1. Discrete measured groupoids and quasi-periodic spaces. Our reference for measured groupoids is [1]. A groupoid G is a set of transformations $\gamma \in G$ between points in a base space X,

$$\gamma : y \to x,$$

where $y = s(\gamma) \in X$ and $x = r(\gamma) \in X$ are respectively the source and the range of γ, subject to the usual algebraic requirements that X is the object space of a small category all of whose morphisms $\gamma \in G$ are invertible. Then $X = \{\gamma \in G : \gamma\gamma = \gamma\} \subset G$, where $s(\gamma) = \gamma^{-1}\gamma$ and $r(\gamma) = \gamma\gamma^{-1}$, and the fibered product

$$G \times_X G = \{(\gamma', \gamma) \in G \times G : r(\gamma) = s(\gamma')\}$$

describes the family of composable transformations. The isomorphism relation between objects of this category is an equivalence relation $R \subset X \times X$ on X (the image of the map $(r, s) : G \to X \times X$). We denote by G^x the set of $\gamma \in G$ such that $r(\gamma) = x$ and G_y the set of $\gamma \in G$ such that $s(\gamma) = y$. A groupoid is called discrete if G^x is a countable set for any $x \in X$.

A **Borel groupoid** is a groupoid G endowed with a standard Borel structure such that the source map and the range map $s, r : G \to X$, the composition map $G \times_X G \to G$ and the inverse map $G \to G$ are Borel, where the Borel structure on X and $G \times_X G$ come from the inclusions in G and $G \times G$. A **discrete measured groupoid** is a discrete Borel groupoid endowed with an invariant measure class on X (i.e., the negligible sets of some probability measure on X). We will call this class **the measure class of** G. Recall that a measure class on X is said to be **invariant** if the saturation of a negligible Borel subset of X is again negligible (where the saturation of $A \subset X$ by G is the union of the ranges of the transformations $\gamma \in G$ with source in A). As G is discrete, the measure class on X gives a canonical measure class on G. A Borel subset $K \subset G$ is negligible if and only if its projection $r(K)$ is negligible in X.

Example 7: Let Γ be a countable group and α be an action of Γ on a standard probability space (X, μ) preserving the measure class of μ. Then the set $G = \amalg_{x \in X} \amalg_{\gamma \in \Gamma} (\gamma x, x)$ is a discrete measured groupoid for the natural Borel structure and the class of μ. It is denoted by $G = \Gamma \ltimes X$. The map $(r, s) : G \to X \times X$ projects G to the equivalence relation $R \subset X \times X$ given by the orbit partition. G coincides with R if and only if α is free.

Let G be a discrete Borel groupoid with base space X. A **fibered Borel space** over X is a standard Borel space Σ together with a surjective Borel projection $r : \Sigma \to X$. The fiber at x is denoted by $\Sigma^x = r^{-1}(x)$. The **reduction** of Σ to a Borel set $A \subset X$ is the measurable fibered space $\Sigma^A = r^{-1}(A)$, over A. A Borel section of Σ is a Borel map $u : X \to \Sigma$ such that $u(x) \in \Sigma^x$ for $x \in X$. A **Borel action** ρ **of** G **on** Σ is a collection of Borel isomorphisms

$$\rho(\gamma) : \Sigma^{s(\gamma)} \to \Sigma^{r(\gamma)},$$

indexed by $\gamma \in G$, such that the map $\rho : G \times_X \Sigma \to \Sigma$ is Borel and satisfy the cocycle condition $\gamma'(\gamma u) = (\gamma'\gamma)u$ for every composable elements $\gamma, \gamma' \in G$, and $\gamma u = u$ when $\gamma \in X$, where we denote $\rho(\gamma, u) = \rho(\gamma)u = \gamma u$. One says that an action is **free** if the equality $\gamma u = u$ implies $\gamma = x$ for every $u \in \Sigma^x$ and every $x \in X$. A **fundamental domain** for the action of G is a Borel subset $F \subset \Sigma$ which contains exactly one point in each G-orbits in Σ. Let now G be a discrete measured groupoid over a probability space (X, μ) and Σ be a Borel fibered space over X. A reduction of Σ to a subset A of X is called **inessential** if A has negligible complement in X. By a measurable action of G on Σ we mean a Borel action of an inessential reduction G^A of G on Σ^A. One defines the notion of (essentially) free actions and (essential) fundamental domains in that context by relaxing the above conditions to inessential reductions. Although the terminology is abusive without irreducibility assumption on G (as ergodicity for example), we will in the present text use the word 'quasi-periodic' in the following sense—compare to [26, 27]. More details on Borel G-spaces can be found in [8, 15].

Definition 8: Let G be a discrete measured groupoid. A G-quasi-periodic standard Borel space is a measurable fibered space Σ endowed with an essentially free measurable action of G with fundamental domain.

In this case we call the projection $r : \Sigma \to X$ the **realization map**. The fibers Σ^x, $x \in X$, are called **realizations** of the quasi-periodic space Σ. Note that G itself is a quasi-periodic standard Borel space, and the realization map coincide with the range map.

3.2. Covolumes on quasi-periodic standard Borel spaces. Let G be a discrete measured groupoid and Σ be a G-quasi-periodic standard Borel space. Denote by $\mathcal{B}_G(\Sigma)$ the σ-algebra of Borel subsets of Σ which are invariant under the action of G (quasi-periodic Borel subsets), and $\mathcal{B}_G^x(\Sigma)$ its trace on the realization Σ^x, so that $\gamma(\mathcal{B}_G^{s(\gamma)}(\Sigma)) = \mathcal{B}_G^{r(\gamma)}(\Sigma)$.

A **Borel system of measures** on a Borel fibered space $(\Sigma, \mathcal{B})$ is a family $(\lambda^x)_{x\in X}$ of measures on Σ such that

- λ^x is supported on the realization Σ^x for every $x \in X$,
- there exists an exhausting sequence of Borel sets $A_n \subset \Sigma$ such that $\lambda^x(A_n) < \infty$ for every $x \in X$ (σ-finiteness),
- for any non-negative Borel map f on Σ, the map $x \mapsto \lambda^x(f)$ is Borel.

A **covolumic system** on a G-quasi-periodic standard Borel space Σ is a Borel system of measure such that λ^x is a measure on the σ-algebra $\mathcal{B}_G^x(\Sigma)$, for every $x \in X$. For instance, given a fundamental domain $F \subset \Sigma$, one can set $\lambda^x(A) = \alpha^x(A \cap F)$, where $(\alpha^x)_{x\in X}$ is Borel system of measures on F.

A **covolume** on Σ is a measure on $(\Sigma, \mathcal{B}_G(\Sigma))$ of the form $\lambda(A) = \int_X \lambda^x(A)d\mu(x)$, where $(\lambda^x)_{x\in X}$ is a covolumic system and μ a measure on X in the class of G. We call the couple (Σ, λ) a G**-quasi-periodic measured space**. The projection $p : \Sigma \to \Sigma/G$ then induces an isomorphism $L^\infty(\Sigma, \lambda) \to L^\infty(\Sigma/G, p_*\lambda)$.

3.3. Isometric representations of discrete groupoids. Let G be a discrete measured groupoid with base space X. A **measurable field of metric spaces** $Y \to X$ over X consists of a metric space (Y^x, d^x) for each $x \in X$ whose total space $Y = \amalg_{x\in X} Y^x$ is endowed with a standard Borel structure such that

- the inclusion $Y^x \subset Y$ is a Borel isomorphism on its image, for every $x \in X$,
- the natural projection $r : Y \to X$ is a Borel map (thus $Y \to X$ is a fibered Borel space),
- the metric $d : Y \times_X Y \to [0, \infty[$ is a Borel map.

An isometric action of G on Y is a Borel action ρ of G on Y such that, up to an inessential reduction, the map $\rho(\gamma) : Y^{s(\gamma)} \to Y^{r(\gamma)}$ is an isometry for any $\gamma \in G$. In this case Y is called a **metric G-space**. We will always assume that Y is **complete** (almost each fiber is complete) and **separable** (there is a Borel set $D \subset Y$ which is countable and dense in restriction to each fiber Y^x).

Examples 9: 1 - Let H be a separable Hilbert space and (X, μ) be a probability space. The product space $\bar{H} = X \times H$ is a measurable field of metric spaces over X in a natural way (for the product Borel structure), and the measurable sections $\xi : X \to \bar{H}$ coincide with that of measurable map $X \to H$. The scalar product on H integrates to a measurable map on $\bar{H} \times_X \bar{H}$ and one recovers in this way the classical notion of measurable field of Hilbert spaces [10], as every such a bundle is trivializable up to dimension considerations. In this case, an isometric action of G is called an affine (isometric) representation of G on $\bar{H}$.

2 - A topological fibered bundle over a polish space is in particular a measurable field of metric spaces (when endowed with a Borel family of metrics along the fiber), usually non-trivializable at the topological level.

3 - Quasi-periodic metric spaces provide interesting classes of metric G-spaces (see [26]).

3.4. Equivariant mappings to metric spaces. Let Σ be a G-quasi-periodic measured space with covolume λ and Y be a metric G-space as above. A measurable fibered map $f : \Sigma \to Y$ is said to be equivariant if

$$f(\gamma u) = \rho(\gamma) f(u)$$

for any $x \in X$, $\gamma \in G_x$ and $u \in \Sigma^x$. For instance, the quasi-periodic functions on Σ are the equivariant functions to $\mathbf{C}$ endowed with the trivial action of G (that is, the action of G on $X \times \mathbf{C}$, with $\rho(\gamma) = 1$ for every $\gamma \in G$). For arbitrary spaces Y, equivariant maps can be constructed by choosing their values on a fundamental domain and extending by equivariance.

One defines the distance between two equivariant maps $f, g : \Sigma \to Y$ by the expression

$$\|f - g\|^2 = \int_\Sigma |f(u) - g(u)|^2 d\lambda(u),$$

and denotes $L^2_G(\Sigma, Y)$ the space of equivariant map which are at bounded distance $\|f - \mathfrak{f}\| < \infty$ from some fixed equivariant $\mathfrak{f} : \Sigma \to Y$, where as usual, we identify two functions if they coincide λ-almost everywhere.

A **complete CAT(0)** G**-space** is a metric G-space whose fibers Y^x are almost surely complete CAT(0) spaces. One denotes by P_2 the space of Borel systems of probability measures on Y with almost everywhere finite second moment. If $\nu \in P_2$, the barycentric codiffusion c^x on Y^x for each $x \in X$ associated to ν^x its center of mass $c^x(\nu^x)$, and the map $c : x \mapsto c^x(\nu^x)$ is Borel. This map is called the **barycentric codiffusion** of the CAT(0) G-space Y.

Lemma 10: *If Y is a complete CAT(0) G-space, then $L^2_G(\Sigma, Y)$ is a complete CAT(0) space as well.*

Proof. Up to an inessential reduction we can assume that each realization is a complete CAT(0) space. The completeness of $L^2_G(\Sigma, Y)$ follows from that of Y as usual. Let $f_0, f_1 \in L^2_G(\Sigma, Y)$. For any $u \in \Sigma^x$ there exists a unique geodesic segment $(f_t(u))_{t \in [0,1]}$ in Y^x from $f_0(u)$ to $f_1(u)$. Then $t \to f_t$ is a geodesic from f_0 to f_1, and given $g \in L^2_G(\Sigma, Y)$, the punctual CAT(0) inequality

$$|g(u) - f_t(u)|^2 \leq (1-t)|g(u) - f_0(u)|^2 + t|g(u) - f_1(u)|^2 - t(1-t)|f_0(u) - f_1(u)|^2$$

integrates with respect to the covolume to give the CAT(0) inequality in $L^2_G(\Sigma, Y)$. ∎

3.5. Diffusions and symmetry of diffusions. Let Σ be a G-quasi-periodic standard Borel space. A (quasi-periodic) **diffusion** on Σ is a map ν which associates to each $u \in \Sigma^x$ a probability measure ν_u on Σ such that,

- ν_u vanishes outside Σ^x,
- for any measurable map

$$f : \Sigma \times_X \Sigma \to \mathbf{R}_+,$$

  the map $u \mapsto \int_\Sigma f(u,v)\nu(u \to v)dv$ is measurable on Σ, where we note $d\nu_u(v) = \nu(u \to v)dv$,
- $\gamma_*(\nu_u) = \nu_v$ for any $\gamma \in G$ such that $\gamma(u) = v$.

Example 11: Assume moreover that the probability measures ν_u have finite support, and denote $\nu(u \to v)$ the probability $\nu_u(v)$ to go from u to v in Σ. Such a diffusion is called a **random walk** on Σ. One can insist on generating properties of ν by assuming that there is only one trajectory in every realization.

We shall say that a diffusion is symmetric with respect to a covolume λ on Σ if it satisfies the following condition, where a map f on $\Sigma \times_X \Sigma$ is called invariant if it is invariant under the diagonal action of G.

Symmetry condition. For any non-negative invariant map f on $\Sigma \times_X \Sigma$,

$$\int_\Sigma \int_\Sigma f(u,v)\nu(u \to v)dvd\lambda(u) = \int_\Sigma \int_\Sigma f(u,v)\nu(v \to u)dud\lambda(v)$$

When the symmetry condition is satisfied, we call the triple (Σ, ν, λ) a ***G*-quasi-periodic diffusion space**. A system which does not diffuse (that is, if $\nu(u \to v)dv = d\delta_u(v)$ is the Dirac measure at $u \in \Sigma$) is obviously symmetric. In what follows we will mainly be interested in diffusion that are both symmetric and generating.

3.6. Energy of equivariant mappings. Let (Σ, ν, λ) be a G-quasi-periodic diffusion space and (Y, d) be a complete metric G-space. Given a measurable equivariant fibered map $f : \Sigma \to Y$ one sets for any $u, v \in \Sigma^x$,

$$df(u)(v) = f(v) - f(u) \in \mathrm{Con}_{f(u)} Y^x$$

the discrete derivative of f, defined (if Y is a CAT(0) space) by projecting $f(v)$ on $\mathrm{Con}_{f(u)}Y^x$, or simply via the local energy map,

$$|df|_\nu^2(u) = \int_\Sigma |df(u)(v)|^2 \nu(u \to v)dv = \int_\Sigma |f(v) - f(u)|^2 \nu(u \to v)dv.$$

The energy of f is

$$E_\nu(f) = \frac{1}{2}\int_\Sigma |df|_\nu^2 d\lambda.$$

When there is only one diffusion we drop the subscript ν in the notation E_ν. (We recall that the tangent cone $\mathrm{Con}_y(Y)$ of a complete CAT(0) space Y at a point $y \in Y$ is the cone over the space $S_y(Y)$ of direction at y, see [5]. This is a complete CAT(0) space as well and there is a distance non-increasing projection $Y \to \mathrm{Con}_y(Y)$ which preserves the distance to y.)

Lemma 12: *The energy function E on $L_G^2(\Sigma, Y)$ is either everywhere finite or everywhere equal to ∞.*

Proof. For $f, g \in L_G^2(\Sigma, Y)$ we have

$$|f(v) - f(u)|^2 \leq 3|f(v) - g(v)|^2 + 3|g(v) - g(u)|^2 + 3|g(u) - f(u)|^2$$

so that $E(f) \leq 3E(g) + 3\|f - g\|^2$, using the symmetry condition. ∎

In order to apply harmonic analysis we shall make from now on the following **finite energy assumption,**

$$E(f) < \infty \quad \text{for any } f \in L_G^2(\Sigma, Y).$$

Note that the convexity of the energy follows immediately from the CAT(0) inequality for non-positively curved target spaces.

Definition 13: A fixed point for the diffusion ν is a mapping $f \in L_G^2(\Sigma, Y)$ which is $\nu(u \to\ )$-almost surely constant for almost all $u \in \Sigma$.

Equivalently a fixed point is a map of zero energy. When the diffusion is "sufficiently generating" (as in Corollaries 1 and 2, for instance), fixed points relative to ν ranges into actual fixed points for the action of G on Y.

3.7. Convolution of diffusions and Poincaré constant. The convolution $\nu * \nu'$ of two symmetric diffusions ν and ν' on (Σ, λ) is the symmetric diffusion defined by the expression

$$\int_\Sigma f(u,v)\nu * \nu'(u \to v)dv = \int_\Sigma \left(\int_\Sigma f(u,v)\nu'(w \to v)dv \right) \nu(u \to w)dw$$

for non-negative invariant f on $\Sigma \times_X \Sigma$. The 2-steps diffusion associated to ν is the diffusion $\nu^2 = \nu * \nu$.

Given a G-quasi-periodic diffusion space Σ and a complete metric G-space Y o ne defines the following **Poincaré constant**

$$\pi_2(f) = E_{\nu^2}(f)/E_\nu(f)$$

for equivariant f and $\pi_2 = \pi_2(\Sigma, Y) = \sup_f \pi_2(f)$ over $f \in L^2_G(\Sigma, Y)$ with non-zero energy. Following [19], we will prove (under additional curvature assumptions on Y yielding convexity of E as well as gradient flows) the existence of fixed points when $\pi_2(\Sigma, Y) < 2$.

4. Heat equation and harmonic analysis

The content of this section is the proof of the fixed point theorem stated in the introduction (Theorem 3). We follow as closely as possible the proof given by Gromov [19, Section 3] in the group case, and we continue to assume finiteness of the energy as described in Section 3.

4.1. Heat operators. Let G be a discrete measured groupoid, (Σ, λ, ν) be a G-quasi-periodic diffusion space and Y be a complete CAT(0) G-space with the barycentric codiffusion c. Denote by $\nu(u \overset{\varepsilon}{\to}\ )$ the ε-mass-transportation from δ_u to $\nu(u \to\ )$,

$$\nu(u \overset{\varepsilon}{\to}\ ) = (1-\varepsilon)\delta_u + \varepsilon\nu(u \to\ ),$$

$\varepsilon \in [0,1]$. The **heat operators** $H^{\overset{\varepsilon}{\to}}$ are defined to be the centers of mass of the push-forward $f_*\nu(u \overset{\varepsilon}{\to}\ )$,

$$H^{\overset{\varepsilon}{\to}} f(u) = c^{r(u)}(f_*\nu(u \overset{\varepsilon}{\to}\ )),$$

for $f \in L^2_G(\Sigma, Y)$.

Lemma 14: *The heat operators $H^{\overset{\varepsilon}{\to}}$ are well-defined on $L^2_G(\Sigma, Y)$.*

Proof. Let $f \in L^2_G(\Sigma, Y)$. As we assumed that $E(f) < \infty$, we get

$$\int |f(u) - f(v)|^2 \nu(u \to v)dv < \infty$$

for a.e. $u \in \Sigma^X$, thus for any $y \in Y^{r(u)}$,

$$\int |y - f(v)|^2 \nu(u \to v) dv \leq 2|y - f(u)|^2 + 2 \int |f(u) - f(v)|^2 \nu(u \to v) dv$$

follows finite as well. A similar computation using 2-convexity gives $H^{\overset{\varepsilon}{\to}} f \in L^2_G(\Sigma, Y)$ where, by L^2 contraction of the transportation metric, we have that $H^{\overset{\varepsilon}{\to}}$ contracts the L^2 metric (and the same for the L^1 metric). ∎

Note that $H^{\overset{1}{\to}} f = f$ if and only if $H^{\overset{\varepsilon}{\to}} f = f$ for any ε by the definition of codiffusions. In this case f is called **harmonic**. There are Laplace operators associated to $H^{\overset{\varepsilon}{\to}}$, which we will not need, and which are defined by diffusing around u and comparing the image by f of this diffusion with the punctual image $f(u)$ (e.g. $\Delta_\varepsilon f(u) = \varepsilon^{-1}(f(u) - H^{\overset{\varepsilon}{\to}} f(u))$ where the difference in taken in $\mathrm{Con}_{f(u)} Y^{p(u)}$ after projection). See [19, page 97] (and page 106 therein).

The behaviour of the heat flow is governed by the following inequality for infinitesimal $\varepsilon \in]0, 1]$, which follows immediately from the averaged 2-convexity inequality (and thus is an equality in the Hilbert space case).

Parabolic growth inequality. Let $x \in X, u \in \Sigma^x$ and $y \in Y^x$. Then,

$$\int_\Sigma |y - f(v)|^2 \nu(u \overset{\varepsilon}{\to} v) dv \geq |y - H^{\overset{\varepsilon}{\to}} f(u)|^2 + \int_\Sigma |H^{\overset{\varepsilon}{\to}} f(u) - f(v)|^2 \nu(u \overset{\varepsilon}{\to} v) dv. \tag{10}$$

4.2. Poincaré estimates and Kazhdan's relaxation constant. Let $\varepsilon \in]0, 1]$ be a fixed real number. For $f \in L^2_G(\Sigma, Y)$ define,

$$\kappa^{\overset{\varepsilon}{\to}}(f) = 1 - (2\varepsilon)^{-1}(1 - E_\nu(H^{\overset{\varepsilon}{\to}} f)/E_\nu(f)),$$

$\kappa^{\overset{\varepsilon}{\to}}(\Sigma, Y) = \sup_f \kappa^{\overset{\varepsilon}{\to}}(f)$ over $f \in L^2_G(\Sigma, Y)$ with non-zero energy, and denote $\kappa = \kappa^{(0)} = \kappa(\Sigma, Y) = \overline{\lim}_{\varepsilon \to 0} \kappa^{\overset{\varepsilon}{\to}}(\Sigma, Y)$. These constants are called the **Kazhdan constants** of the diffusion ν [19, page 114]. Recall that the Poincaré constant π_2 of ν is defined at the end of the previous section.

Lemma 15: *Given $f \in L^2_G(\Sigma, Y)$ and $u \in \Sigma$ one has*

$$|H^{\overset{\varepsilon}{\to}} f(u) - f(u)|^2 \leq \varepsilon \int |f(u) - f(v)|^2 \nu(u \to v) dv$$

and

$$|H^{\overset{\varepsilon}{\rightarrow}} f(u) - f(u)| \leq \varepsilon \int |f(u) - f(w)| \nu(u \to w) dw.$$

In particular $\|H^{\overset{\varepsilon}{\rightarrow}} f - f\|^2 \leq \varepsilon E(f)$.

Proof. This follows from the transportation inequalities (7) and (9). ∎

Proposition 16: *For any CAT(0) space* Y *one has* $\pi_2 \geqslant 1 + \kappa$.

Proof. Let $f \in L^2_G(\Sigma, Y)$ be a function with non-zero energy. Taking $y = H^{\overset{\varepsilon}{\rightarrow}} f(w)$ in the parabolic growth inequality we have

$$\int_\Sigma |H^{\overset{\varepsilon}{\rightarrow}} f(w) - f(v)|^2 \nu(u \overset{\varepsilon}{\rightarrow} v) dv \geq |H^{\overset{\varepsilon}{\rightarrow}} f(w) - H^{\overset{\varepsilon}{\rightarrow}} f(u)|^2 + \int_\Sigma |H^{\overset{\varepsilon}{\rightarrow}} f(u) - f(v)|^2 \nu(u \overset{\varepsilon}{\rightarrow} v) dv.$$

Let us integrate this inequality with respect to $\iint \cdots \nu(u \to w) dw d\lambda(u)$. Using the symmetry hypothesis on ν this gives

$$\int (1-\varepsilon) \int |H^{\overset{\varepsilon}{\rightarrow}} f(u) - f(v)|^2 \nu(u \to v) dv + \varepsilon \int |H^{\overset{\varepsilon}{\rightarrow}} f(u) - f(v)|^2 \nu^2(u \to v) dv d\lambda(u)$$
$$\geq 2E_\nu(H^{\overset{\varepsilon}{\rightarrow}} f) + (1-\varepsilon)\|f - H^{\overset{\varepsilon}{\rightarrow}} f\|^2 + \varepsilon \iint |H^{\overset{\varepsilon}{\rightarrow}} f(u) - f(v)|^2 \nu(u \to v) dv d\lambda(u).$$

Similarly we now take $y = f(w)$ in the parabolic growth inequality, giving

$$\int_\Sigma |f(w) - f(v)|^2 \nu(u \overset{\varepsilon}{\rightarrow} v) dv \geq |f(w) - H^{\overset{\varepsilon}{\rightarrow}} f(u)|^2 + \int_\Sigma |H^{\overset{\varepsilon}{\rightarrow}} f(u) - f(v)|^2 \nu(u \overset{\varepsilon}{\rightarrow} v) dv,$$

and integrate with respect to $\iint \cdots \nu(u \to w) dw d\lambda(u)$:

$$2(1-\varepsilon) E_\nu(f) + 2\varepsilon E_{\nu^2}(f) \geq \iint |H^{\overset{\varepsilon}{\rightarrow}} f(u) - f(v)|^2 \nu(u \to v) dv d\lambda(u) + (1-\varepsilon)\|f - H^{\overset{\varepsilon}{\rightarrow}} f\|^2 + \varepsilon \iint |H^{\overset{\varepsilon}{\rightarrow}} f(u) - f(v)|^2 \nu(u \to v) dv d\lambda(u).$$

Let us add the two inequalities just obtained. We get

$$\begin{aligned}(11)\quad & \frac{1}{2}\iint |H^{\overset{\varepsilon}{\to}} f(u)-f(v)|^2\nu^2(u\to v)dvd\lambda(u)\\ &\quad -\iint |H^{\overset{\varepsilon}{\to}} f(u)-f(v)|^2\nu(u\to v)dvd\lambda(u)+E_{\nu^2}(f)-E_\nu(f)\\ &\qquad -\frac{1}{2}\iint |H^{\overset{\varepsilon}{\to}} f(u)-f(v)|^2\nu(u\to v)dvd\lambda(u)\\ &\qquad\quad \geq \varepsilon^{-1}(1-\varepsilon)\|f-H^{\overset{\varepsilon}{\to}} f\|^2+\varepsilon^{-1}(E_\nu(H^{\overset{\varepsilon}{\to}} f)-E_\nu(f)).\end{aligned}$$

On the other hand we have the following estimate

$$\begin{aligned}&\left|\iint |H^{\overset{\varepsilon}{\to}} f(u)-f(v)|^2\nu(u\to v)dvd\lambda(u)-2E_\nu(f)\right|\\ &\quad\leq \iint ||H^{\overset{\varepsilon}{\to}} f(u)-f(v)|^2-|f(u)-f(v)|^2|\nu(u\to v)dvd\lambda(u)\\ &\quad\leq \iint |H^{\overset{\varepsilon}{\to}} f(u)-f(u)|(|H^{\overset{\varepsilon}{\to}} f(u)-f(v)|\\ &\qquad +|f(u)-f(v)|)\nu(u\to v)dvd\lambda(u)\\ &\quad\leq \iint |H^{\overset{\varepsilon}{\to}} f(u)-f(u)|^2\\ &\qquad +2|H^{\overset{\varepsilon}{\to}} f(u)-f(u)||f(u)-f(v)|\nu(u\to v)dvd\lambda(u)\end{aligned}$$

which gives, taking into account Lemma 15,

$$\begin{aligned}&\left|\iint |H^{\overset{\varepsilon}{\to}} f(u)-f(v)|^2\nu(u\to v)dvd\lambda(u)-2E_\nu(f)\right|\\ (12)\quad &\quad\leq 2\varepsilon E_\nu(f)+\varepsilon\iint |f(u)-f(w)|^2\nu(u\to w)dwd\lambda(u)\\ &\qquad +\varepsilon\iint |f(u)-f(v)|^2\nu(u\to v)dvd\lambda(u)=6\varepsilon E_\nu(f).\end{aligned}$$

Now by combining (11) and (12) we get

$$\begin{aligned}&2E_{\nu^2}(f)-4E_\nu(f)+3\varepsilon E_{\nu^2}(f)+9\varepsilon E_\nu(f)\\ &\qquad\geq \varepsilon^{-1}(1-\varepsilon)\|f-H^{\overset{\varepsilon}{\to}} f\|^2+\varepsilon^{-1}(E_\nu(H^{\overset{\varepsilon}{\to}} f)-E_\nu(f)),\end{aligned}$$

which, after dividing everything by $E_\nu(f)$ and ignoring the term $\|f-H^{\overset{\varepsilon}{\to}} f\|^2$, gives,

$$\pi_2(f)-2+\frac{3}{2}\varepsilon(\pi_2(f)+3)\geq \kappa^{\overset{\varepsilon}{\to}}(f)-1.$$

Hence the result. ∎

Proposition 17: *If $\kappa(\Sigma, Y) < 1$, then the diffusion ν has a fixed point.*

Proof. Take ε so that $\kappa^{\overset{\varepsilon}{\to}} < 1$ and $f_0 \in L^2_G(\Sigma, Y)$. Put $f_i = H^{\overset{\varepsilon}{\to}} f_{i-1}$. If one of the f_i has zero energy we are done. If not, one has

$$E(f_{i+1}) \leq (1 - 2\varepsilon(1 - \kappa^{\overset{\varepsilon}{\to}}))E(f_i)$$

by definition of $\kappa^{\overset{\varepsilon}{\to}}$, so that $\sum_i E(f_i) < \infty$ by assumption. As $\|H^{\overset{\varepsilon}{\to}} f - f\|^2 \leq \varepsilon E(f)$, the sequence (f_i) is Cauchy and thus converge to $f_\infty \in L^2_G(\Sigma, Y)$. By continuity of E we have $E(f_\infty) = 0$, so f_∞ is a fixed point. ∎

Theorem 3 follows from Proposition 16 and Proposition 17. See also [19, page 125].

5. Poincaré inequalities via integral geometry

5.1. Random walks on finite graphs. Given a finite graph L and a metric space Y one defines $\lambda_1(L, Y)$ to be the largest constant λ such that the following Poincaré inequality

$$\frac{1}{2\tau} \sum_{v,w \in L^{(0)}} |f(u) - f(v)|^2 \tau(u)\tau(v) \leq \frac{1}{\lambda} \sum_{(v,w) \in L^{(1)}} |f(u) - f(v)|^2$$

holds true for any map $f : L^{(0)} \to Y$, where $\tau(u)$ is the valence of a vertex $u \in L^{(0)}$, τ is the number of edges in L. When $Y = \mathbf{R}$ is the real line one denotes $\lambda_1(L, \mathbf{R}) = \lambda_1(L)$. Thus $\lambda_1(L, H) = \lambda_1(L)$ for any Hilbert space H (by taking coordinates), while for more general CAT(0) spaces Y one obviously has $\lambda_1(L, Y) \leq \lambda_1(L)$ as any such a Y contains a geodesic line.

5.2. Random walks on quasi-periodic simplicial complexes. Let G be a discrete measured groupoid, and Σ be a G-quasi-periodic simplicial complex of dimension 2. We assume almost every realization of Σ to be connected. As the vertex set $\Sigma^{(0)}$ of Σ is countable for each realization, the measure class on G determine, via the projection $r : \Sigma^{(0)} \to X$, a canonical measure class on $\Sigma^{(0)}$ supported on quasi-periodic Borel sets. By a 0-covolume on Σ we mean a covolume μ which is in this measure class. We assume that Σ has finite covolume, i.e. $\int_\Sigma c(u) d\mu(u) < \infty$ where $c(u)$ is the number of simplexes attached to $u \in \Sigma$.

Consider the fibered product $\Sigma^{(0)} \times_X \Sigma^{(0)}$ endowed with the measure class on quasi-periodic subsets coming from G as above. There are two canonical measures in this class, the vertical counting measure $\int \sum_v f(u,v) d\mu(u)$ and the horizontal one $\int \sum_u f(u,v) d\mu(v)$, for non-negative invariant f. Denoting $\delta(u,v)$ the Radon-Nikodỳm derivative we have

$$\int \sum_v f(u,v) d\mu(u) = \int \sum_u f(u,v) \delta(u,v) d\mu(v)$$

for non-negative invariant f.

Let ν be a random walk on Σ, that is a diffusion whose probability measures $\nu(u \to \ )$ are supported on the set of vertices of the realisation containing u, such that the probability $\nu(u \to v)$ to go from u to v is not zero if and only if (u,v) is an edge of Σ. We assume that, almost surely,

$$\nu(u \to v)\sqrt{\delta(v,u)} = \nu(v \to u)\sqrt{\delta(u,v)}.$$

Note that this symmetry condition (compare to [28]) also reads

$$\int \sum_v f(u,v) \nu(u \to v) d\mu(u) = \int \sum_u f(u,v) \nu(v \to u) d\mu(v)$$

for non-negative invariant f.

5.3. Integral geometry. See [19, pages 125–127] and the “extra-remarks” on page 128.

Theorem 18: *Let G be a discrete measured groupoid and Σ be a G-quasi-periodic simplicial complex of dimension 2 and finite covolume with respect to μ. Assume that there is a real number $\delta_\mu \geqslant 1$ such that $\delta_\mu^{-1} \leqslant \delta(u,v) \leqslant \delta_\mu$ for almost all edges (u,v) in $\Sigma^{(1)}$ and that almost each link L of Σ is connected. Let Y be a field of CAT(0) spaces such that*

$$\lambda_1(L, Y^x) \geqslant \lambda > \delta_\mu^3/2$$

for a real number λ independent of L and x. Then for any isometric action of G on Y and any equivariant map $\mathfrak{f} : \Sigma \to Y$ with

$$\int_\Sigma \sum_{(u,v) \in \Sigma^{(1)}} |\mathfrak{f}(u) - \mathfrak{f}(v)|^2 d\mu(u) < \infty,$$

there is an equivariant map $f : \Sigma \to Y$ at finite L^2-distance from $\mathfrak{f}$ which is constant on almost each realization. In particular, G has a fixed point in Y provided such a map $\mathfrak{f}$ exists.

Proof. Denote by L_u the link of a vertex u of Σ, $\tau(u,v)$ the number of triangles in Σ containing the edge $(u,v) \in \Sigma^{(1)}$, and $\tau(u)$ the number of triangles attached to u. Let $f \in L^2_G(\Sigma, Y)$, where $L^2_G(\Sigma, Y)$ is centered at $\mathfrak{f}$. By hypothesis we have

$$\frac{1}{2\tau(u)} \sum_{v,w \in L_u^{(0)}} |f(w) - f(v)|^2 \tau(u,v)\tau(u,w) \leq \frac{1}{\lambda} \sum_{(v,w) \in L_u^{(1)}} |f(w) - f(v)|^2.$$

Let's integrate this inequalities.

$$\begin{aligned} \int_\Sigma \sum_{(v,w)\in L_u^{(1)}} & |f(w) - f(v)|^2 d\mu(u) \\ &= \int_\Sigma \sum_{(v,w)\in\Sigma} \sum_{(u,v,w)\in\Sigma} |f(w) - f(v)|^2 \delta(u,v) d\mu(v) \\ &= \int_\Sigma \sum_{(v,w)\in\Sigma} |f(w) - f(v)|^2 \tau_\delta(v,w) d\mu(v) \end{aligned}$$

where

$$\tau_\delta(v,w) = \sum_{(u,v,w)\in\Sigma} \delta(u,v)$$

and

$$\begin{aligned} \int_\Sigma \frac{1}{2\tau(u)} & \sum_{v,w\in L_u^{(0)}} |f(w) - f(v)|^2 \tau(u,v)\tau(u,w) d\mu(u) \\ &= \int_\Sigma \sum_{w\in\Sigma^v} |f(w) - f(v)|^2 \sum_{L_u^{(0)} \ni v,w} \frac{1}{2\tau(u)} \tau(u,v)\tau(u,w)\delta(u,v) d\mu(v) \\ &= \int_\Sigma \sum_{w\in\Sigma^u} |f(w) - f(v)|^2 \underline{\tau}_\delta(v,w) d\mu(v) \end{aligned}$$

where

$$\underline{\tau}_\delta(v,w) = \sum_{L_u^{(0)} \ni v,w} \frac{1}{2\tau(u)} \tau(u,v)\tau(u,w)\delta(u,v).$$

Define

$$\tau_\delta(v) = \frac{1}{2} \sum_{(u,v)\in\Sigma} \delta(u,v)\tau(u,v).$$

One has

$$\sum_{w\in\Sigma^v} \tau_\delta(v,w) = \sum_{w\in\Sigma^v} \underline{\tau}_\delta(v,w) = 2\tau_\delta(v),$$

so that, putting

$$\nu(v \to w) = \frac{\tau_\delta(v,w)}{2\tau_\delta(v)}$$

and

$$\underline{\nu}(v \to w) = \frac{\underline{\tau}_\delta(v,w)}{2\tau_\delta(v)}$$

we get a Poincaré inequality,

$$\int_\Sigma \sum_w |f(w)-f(v)|^2 \underline{\nu}(v \to w) d\mu'(v) \leq \frac{1}{\lambda} \int_\Sigma \sum_w |f(w)-f(v)|^2 \nu(v \to w) d\mu'(v)$$

where $d\mu'(v) = 2\tau_\delta(v) d\mu(v)$. Let us note that μ' is symmetric relatively to ν et $\underline{\nu}$. Then we have

$$\nu^2(v \to w) = \sum_{L_u^{(0)} \ni v,w} \frac{\tau_\delta(v,u)}{2\tau_\delta(v)} \frac{\tau_\delta(u,w)}{2\tau_\delta(u)}.$$

By the hypothesis $\tau_\delta(u,v) \leqslant \delta_\mu \tau(u,v)$, $\tau_\delta(u,w) \leqslant \delta_\mu \tau(u,w)$ and $\tau_\delta(u) \geqslant \tau(u)/\delta_\mu$; so

$$\nu^2(v \to w) \leqslant \delta_\mu^3 \underline{\nu}(v \to w)$$

and we get the following Poincaré inequality

$$E_{\nu^2}(f) \leqslant \frac{\delta_\mu^3}{\lambda} E_\nu(f).$$

Thus the diffusion ν has a fixed point in Y provided that $\lambda > \delta_\mu^3/2$. ∎

Remarks 19: 1. - It would be interesting to obtain a similar result without the finite energy assumption.

2. - Working with Wang's definition [32] of the constant λ_1 immediately leads to the inequality $\frac{1}{2}\int_\Sigma \sum_w |H^{\overset{1}{\to}} f(w) - f(v)|^2 \nu(v \to w) d\mu'(v) \leq \pi \delta_\mu E_\nu(f)$, with the notations of the proof, showing in particular the absence of non-constant harmonic maps in $L^2_G(\Sigma, Y)$ under the local assumption $\pi\delta_\mu < 1$. Depending on the situation, it may be possible then to show directly the existence of harmonic mappings (thus proving the fixed point theorem), for instance under local compactness assumptions on the target space or via scaling limits, see [31, 32], [19, pages 107 and 127]. See also [21]. The advantage of Wang's definition is that it is controllable in terms of the tangent cones of the target space; for instance its computation for Hadamard manifolds reduces to the case of the real line.

3. - If the action on Y has a fixed point y, then for every map $f : \Sigma \to Y$ at L^2 distance 1 from this fixed point with sufficiently low energy, the heat flow starting at f converges to a fixed point different from y (cf. Proposition 17). For Y a field of Hilbert spaces this implies Kazhdan's property T (compare [17]), but general (fields of) CAT(0) spaces, including some affine buildings, may have small isotropy group(oid)s.

6. Harmonic analysis and foliated rigidity

6.1. Proof of Corollary 1: rigidity of projections into non-positively curved laminations.

Proof. Let Σ be the measurable field of simplicial complexes associated to the triangulation K (the 1-skeleton of the fiber Σ^x at $x \in X$ is the graph of the class of x, and its 2-dimensional simplicial structure is given by attaching a triangle to each 3-cycle in K). This gives a R-quasi-periodic simplicial complex Σ with finite covolume.

Define Y' to be the field of manifolds whose fiber at $p \in X'$ is the leaf ℓ_p passing through p, endowed with the structure of measurable field of metric spaces coming from a foliated atlas of M. Then Y' is endowed with an obvious isometric action of R' which, when pulled back t o X, give an isometric action of R on the measurable field of manifolds Y whose fiber at $x \in X$ is the leaf passing through $\pi(x)$. The map π extends to an equivariant Borel mapping q from Y to Y', defined by $q(x, y) = (\pi(x), y)$ for $(x, y) \in Y$. Let ρ be the projection from Y' to Y'/R'. Note that Y'/R' is a Borel subset of F.

View the homomorphism π as a section of Y and extend it by equivariance to a map $\mathfrak{f} : \Sigma \to Y$ (hence $\mathfrak{f}$ is zero on $\Sigma \backslash \Sigma^{(0)}$). Let $L^2_R(\Sigma, Y)$ be the space of equivariant maps from Σ to Y which are at finite distance from $\mathfrak{f}$. By our assumptions $\lambda_1(L, Y^x) \geq \lambda > 1/2$ and $\mathfrak{f}$ has finite energy so harmonic analysis applies (Theorem 18) to give an equivariant map $\xi \in L^2_R(\Sigma, Y)$ at finite distance from $\mathfrak{f}$ and constant on the fibers of Σ. In particular ξ descends to a measurable section $\underline{\xi} : X \to Y$. In turn this gives us a measurable map $\eta = \rho \circ q \circ \underline{\xi} : X \to Y'/R'$ such that $\eta(x) \in \ell_{\pi(x)}$ almost surely. It follows from equivariance of ξ that η is constant on almost every orbit of R. As μ is ergodic, this implies that η is essentially constant on X. The essential image of π is then a countable subset of the leaf containing the essential image of η. ∎

6.2. Proof of Corollary 2: orbit equivalence rigidity.

Proof. Note that the assumptions imply that R has Kazhdan's property T, see [28]. Let Σ be the 2-dimensional quasi-periodic simplicial complex associated to K as above, with fundamental domain F and finite covolume μ. Consider an action ρ of Γ on Y, and the associated action of R' on $X'\times Y$. Pull it back to an action of R on $X\times Y$, fix a point $y_0\in Y$, and denote by $\mathfrak{f}$ the equivariant map from Σ to $X\times Y$ whose value on F is y_0. Let $L^2_R(\Sigma,Y)$ be the space of equivariant map from Σ to $X\times Y$ which are at finite distance from $\mathfrak{f}$. We have

$$\begin{aligned}\int_\Sigma \sum_{(u,v)\in\Sigma^{(1)}} |\mathfrak{f}(u)-\mathfrak{f}(v)|^2 d\mu(u) &= \int_\Sigma \sum_{(u,v)\in\Sigma^{(1)}} |y_0 - c(\pi(u,v))y_0|^2 d\mu(u)\\ &\leq C^2 \int_\Sigma \sum_{(u,v)\in\Sigma^{(1)}} |c(\pi(u,v))|^2_\Gamma d\mu(u) < \infty,\end{aligned}$$

where the metric in Γ is relative to a finite symmetric generating set S, and $C=\sup_{s\in S}|y_0-\rho(s)y_0|$. Thus there is an equivariant map $f\in L^2_R(\Sigma,Y)$ which is constant on the fibers of Σ. Then

$$\begin{aligned}\int_X |y_0-f(x)|^2 d\mu_X(x) &\leq \int_X \sum_{u\in F^x} |y_0-f(u)|^2 d\mu_X(x)\\ &= \int_\Sigma |\mathfrak{f}(u)-f(u)|^2 d\mu(u) < \infty,\end{aligned}$$

so that $f_*\mu_X\in P_2(Y)$ admits a center of mass $z\in Y$. For $\gamma\in\Gamma$ and $y\in Y$, we have

$$\int_X |\gamma y - f(x)|^2 d\mu_X(x) = \int_X |y-f(\gamma x)|^2 d\mu_X(x) = \int_X |y-f(x)|^2 d\mu_X(x)$$

by invariance of μ_X. This shows that z is a fixed point for the action of Γ. ∎

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